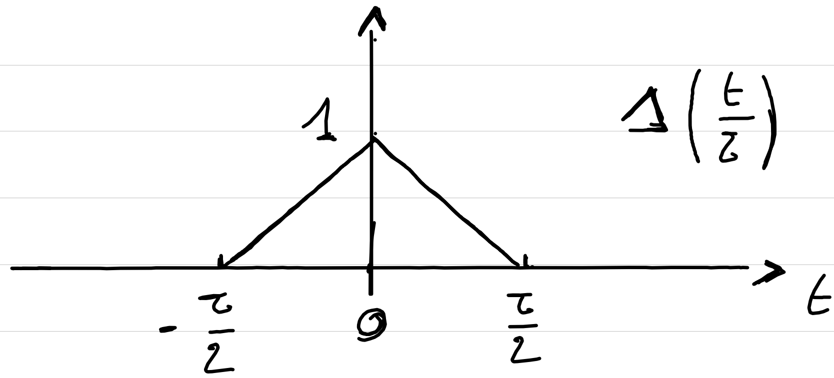
Fourier Transform of a Triangular pulse



$$\Delta\left(\frac{t}{\tau}\right) = \begin{cases} 1 - \frac{2t}{\tau} & \text{for } |t| \leq \frac{\tau}{2} \\ 0 & \text{otherwise} \end{cases}$$

Let's call $x(t) = \Delta\left(\frac{t}{\tau}\right)$ and

$\mathcal{F}[x(t)] = X(\omega)$. (Fourier Transform)

$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-i\omega t} dt$ but since x is not null in $\left[-\frac{\tau}{2}, \frac{\tau}{2}\right]$

$$X(\omega) = \underbrace{\int_{-\frac{\tau}{2}}^{0} \left(1 + \frac{2t}{\tau}\right) e^{-i\omega t} dt}_{A} + \underbrace{\int_{0}^{\frac{\tau}{2}} \left(1 - \frac{2t}{\tau}\right) e^{-i\omega t} dt}_{B}.$$

For A let's change the variable t such as $t' = -t$. So $t = 0 \rightarrow t' = 0$

and $t = -\frac{\tau}{2} \rightarrow t' = +\frac{\tau}{2}$, $dt = -dt'$

and $e^{-i\omega t} = e^{i\omega t'}$

So $A = \int_{0}^{\frac{\tau}{2}} \left(1 - \frac{2t'}{\tau}\right) e^{i\omega t'} dt'$ and

$$X(\omega) = \int_0^{\frac{\tau}{2}} \left(1 - \frac{2t}{\tau}\right) e^{i\omega t}\, dt + \int_0^{\frac{\tau}{2}} \left(1 - \frac{2t}{\tau}\right) e^{-i\omega t}\, dt.$$

$$X(\omega) = \int_0^{\frac{\tau}{2}} e^{i\omega t}\, dt + \int_0^{\frac{\tau}{2}} e^{-i\omega t}\, dt - \frac{2}{\tau}\int_0^{\frac{\tau}{2}} t\left(e^{i\omega t} + e^{-i\omega t}\right) dt,$$

$$X(\omega) = \int_0^{\frac{\tau}{2}} \underbrace{\left(e^{i\omega t} + e^{-i\omega t}\right)}_{2\cos \omega t} dt - \frac{2}{\tau}\int_0^{\frac{\tau}{2}} t\underbrace{\left(e^{i\omega t} + e^{-i\omega t}\right)}_{2\cos \omega t} dt.$$

See Euler's Formula

And $X(\omega) = \frac{2}{\omega}\left[\sin \omega t\right]_0^{\frac{\tau}{2}} - \frac{4}{\tau} \underbrace{\int_0^{\frac{\tau}{2}} t \cos \omega t\, dt}_{C}$

We can compute C with integral by parts

$$\int u v' = [uv] - \int u' v$$

if $u = t \rightarrow u' = 1$
$v' = \cos \omega t \rightarrow v = \frac{\sin \omega t}{\omega}$

So $C = \frac{1}{\omega}\left[t \sin \omega t\right]_0^{\frac{\tau}{2}} - \frac{1}{\omega}\int_0^{\frac{\tau}{2}} \sin \omega t\, dt$

and $C = \frac{\tau}{2\omega}\sin\left(\frac{\omega\tau}{2}\right) + \frac{1}{\omega^2}\left[\cos \omega t\right]_0^{\frac{\tau}{2}}.$

So going back to $X(\omega)$ we have

$$X(\omega) = \frac{2}{\omega}\left(\sin\left(\frac{\omega\tau}{2}\right)\right) - \frac{4}{\tau}\frac{\tau}{2\omega}\sin\left(\frac{\omega\tau}{2}\right) - \frac{4}{\tau}\frac{1}{\omega^2}\left(\cos\left(\frac{\omega\tau}{2}\right) - 1\right).$$

$$X(\omega) = \cancel{\frac{2}{\omega}\sin\left(\frac{\omega\tau}{2}\right)} - \cancel{\frac{2}{\omega}\sin\left(\frac{\omega\tau}{2}\right)} + \frac{4}{\omega^2\tau}\left(1 - \cos\left(\frac{\omega\tau}{2}\right)\right)$$

and $X(\omega) = \frac{4}{\omega^2\tau}\left(1 - \cos\left(\frac{\omega\tau}{2}\right)\right)$,

$$X(\omega) = \frac{8}{\omega^2\tau}\left(\frac{1 - \cos\left(2\omega\frac{\tau}{4}\right)}{2}\right)$$ with few manips

and we know that $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

imagine that $\theta = \frac{\omega\tau}{4}$ this is what we have

and finally $X(\omega) = \frac{8}{\omega^2\tau}\sin^2\left(\frac{\omega\tau}{4}\right)$ which

be written

$$X(\omega) = \frac{\tau}{2}\,\frac{\sin^2\left(\frac{\omega\tau}{4}\right)}{\left(\frac{\omega\tau}{4}\right)^2}$$

$$X(\omega) = \frac{\tau}{2}\,\mathrm{sinc}^2\left(\frac{\omega\tau}{4}\right)$$

$$\Delta\left(\frac{t}{\tau}\right) \xrightarrow{FT} \frac{\tau}{2}\,\mathrm{sinc}^2\left(\frac{\omega\tau}{4}\right).$$

The Fourier Transform of a triangular pulse is the square of a sine cardinal.