Nantes Université
Department of Earth Sciences
http://www.univ-nantes.fr/sciences/dptstu

The Error Function, often denoted as $\mathrm{erf}(x)$, is a special mathematical function that arises frequently in probability, statistics, and partial differential equations, particularly in problems involving the normal distribution and diffusion processes.

The Error Function is defined by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt, \tag{1}$$

where $x$ is a real number. This function is plotted in Fig. 1.

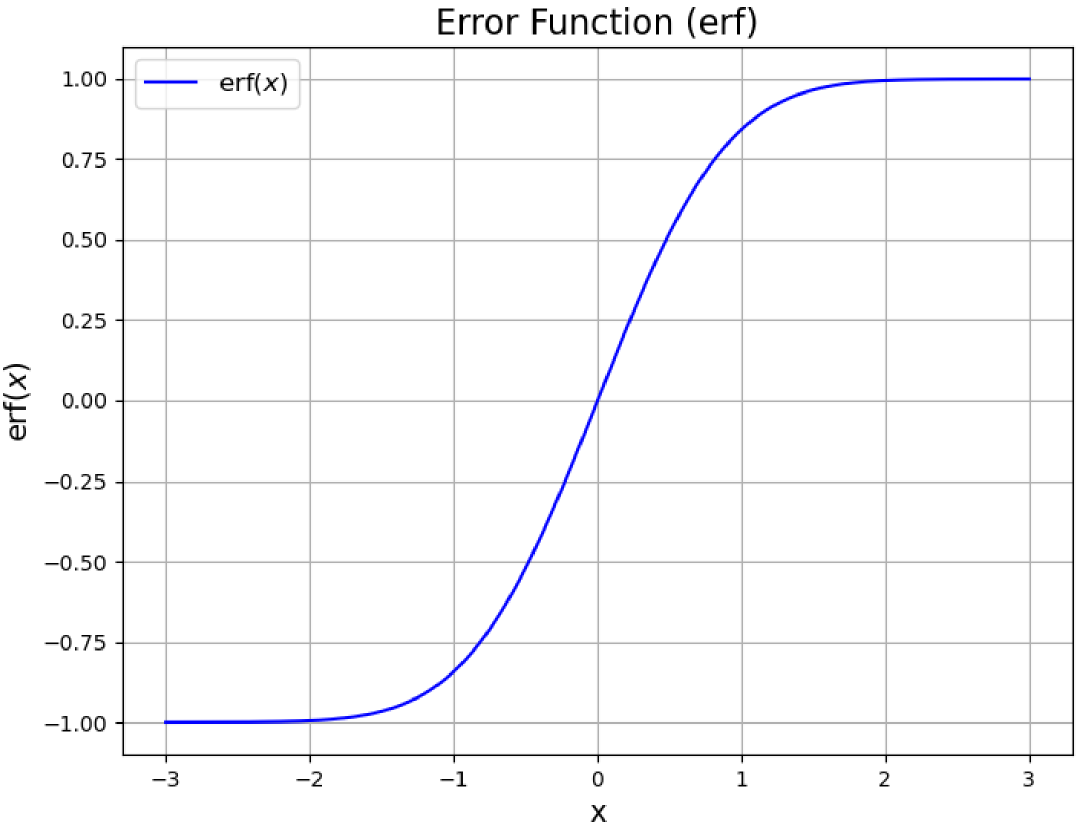


Figure 1: The Error Function of a real variable $x$.

## Properties of the Error Function

- **Symmetry:** The Error Function is odd, meaning that

$$\mathrm{erf}(-x) = -\mathrm{erf}(x).$$

- **Limits:** As $x \to \infty$ or $x \to -\infty$, the function approaches

$$\mathrm{erf}(\infty) = 1 \quad \text{and} \quad \mathrm{erf}(-\infty) = -1.$$

- **Special Values:**

$$\mathrm{erf}(0) = 0, \quad \mathrm{erf}(1) \approx 0.8427 \quad \text{and} \quad \mathrm{erf}(-1) \approx -0.8427.$$

(rev. 1.4)

- **Relationship to the CDF of the Normal Distribution:**

$$\Phi(x) = \frac{1}{2}\left[1 + \text{erf}\left(\frac{x}{\sqrt{2}}\right)\right],$$

where $\Phi(x)$ is the cumulative distribution function (CDF) of the standard normal distribution.

### Complementary Error Function

The complementary Error Function, denoted $\text{erfc}(x)$, is

$$\text{erfc}(x) = 1 - \text{erf}(x). \tag{2}$$

This is useful for tail probabilities and asymptotic approximations.

### Approximation and Series Expansion

For small values of $x$, the Error Function can be approximated using a Taylor series expansion,

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3} + \frac{x^5}{10} - \cdots\right).$$

For large values of $x$, asymptotic expansions are used for efficient computation.

Finally, The Error Function is:

- closely related to the normal distribution's CDF and is widely used in probability theory.
- It describes solutions to the heat equation in diffusion processes.
- and it is Used in signal processing for analyzing Gaussian noise.

## The Error Function in Heat Diffusion

The Error Function is closely tied to heat diffusion because it naturally arises as the solution to the heat equation (also known as the diffusion equation) in specific boundary conditions. Let's explore why and in which case the Error Function is used.

### Heat Diffusion and the Error Function

The heat equation models how heat spreads through a medium over time. In one dimension, the heat equation is

$$\frac{\partial u(x,t)}{\partial t} = \alpha \frac{\partial^2 u(x,t)}{\partial x^2},$$

where:

- $u(x, t)$ is the temperature at position $x$ and time $t$,
- $\alpha$ is the thermal diffusivity of the material,
- $\frac{\partial u(x,t)}{\partial t}$ is the rate of change of temperature with time, and
- $\frac{\partial^2 u(x,t)}{\partial x^2}$ is the spatial second derivative.

### The Semi-Infinite Solid Case

The Error Function typically appears in the solution to the heat equation for a semi-infinite solid—a case where heat diffuses into a material that is infinite in extent in one direction but bounded at the surface. This can represent, for instance, heat diffusing into the ground or a solid block of material with one exposed surface.

### Problem Setup

Consider the following scenario:

- At time $t = 0$, a solid material (occupying $x \geq 0$) has an initial uniform temperature $u_0$.
- Suddenly, at $t = 0$, the surface at $x = 0$ is brought to a new constant temperature $u_s$ (e.g., by immersion in a heat bath or contact with a heat source).
- Heat will then diffuse from the surface into the solid over time.

### Boundary and Initial Conditions

- **Initial condition:** $u(x, 0) = u_0$ for all $x \geq 0$ (the entire solid starts at temperature $u_0$).
- **Boundary condition:** $u(0, t) = u_s$ for $t > 0$ (the surface temperature is fixed at $u_s$).
- As $x \to \infty$, $u(x, t) \to u_0$, meaning that far away from the surface, the temperature remains at the initial temperature since the heat has not yet diffused there.

### Solution Involving the Error Function

The solution to this problem, representing the temperature $u(x, t)$ at any depth $x$ and time $t$, is

$$u(x, t) = u_s + (u_0 - u_s)\, \mathrm{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right). \tag{3}$$

Here, the Error Function $\mathrm{erf}(z)$ appears as a direct consequence of solving the heat equation with the given boundary conditions.

### Why the Error Function?

The Error Function arises because the diffusion process involves integrating a Gaussian function, which naturally appears in the solution of the heat equation for problems involving sudden changes in boundary conditions. The Gaussian profile represents how heat spreads from the boundary into the material, with the Error Function describing the cumulative effect of this diffusion. The Error Function represents the integral of the Gaussian, providing a cumulative view of how heat penetrates the material.

### Interpretation

- At $x = 0$, the Error Function is $\text{erf}(0) = 0$, so $u(0, t) = u_s$, matching the boundary condition that the surface temperature is fixed at $u_s$.
- As $x \to \infty$, the Error Function tends to $\text{erf}(\infty) = 1$, meaning that deep into the material, the temperature remains close to the initial temperature $u_0$.
- For intermediate $x$ and $t$, the Error Function describes the gradual transition between the surface temperature and the material's initial temperature as heat diffuses inward.

The Error Function is used in heat diffusion problems involving:

- Semi-infinite solids (materials with one boundary exposed to a sudden change in temperature).
- Sudden temperature changes at the boundary (such as sudden heating or cooling of the surface).
- One-dimensional heat flow where heat diffuses in only one direction, typically into the material from its surface.

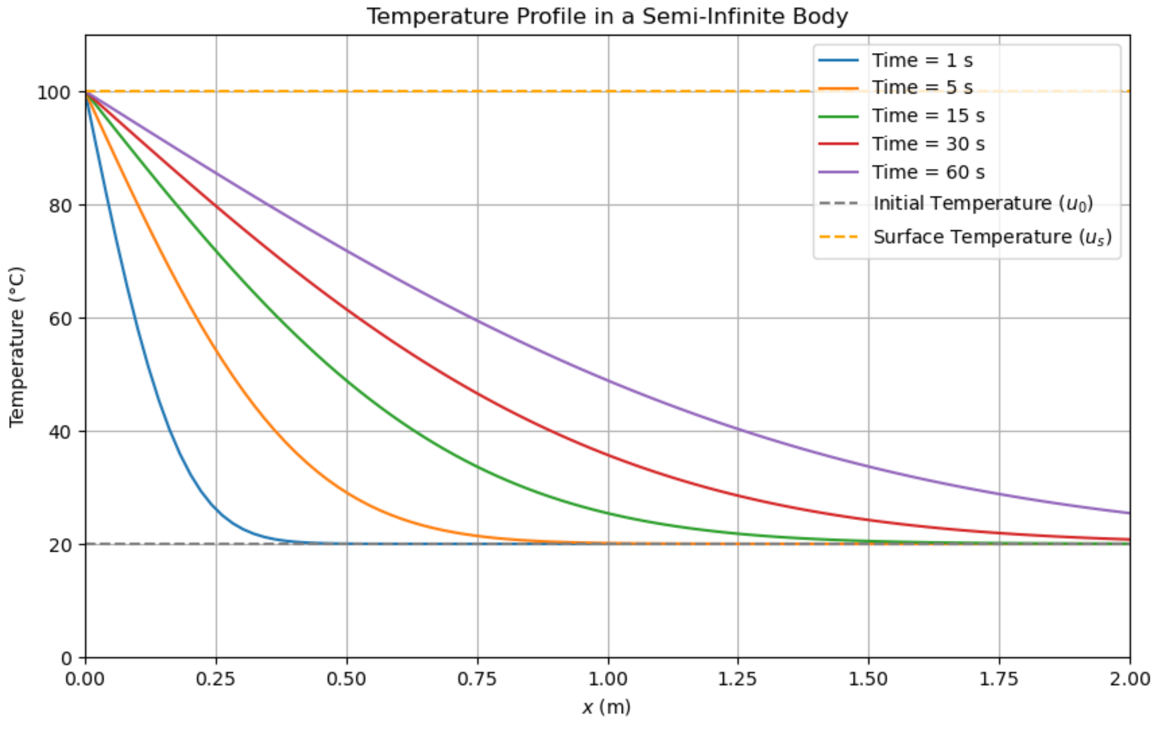


Figure 2: Temperature profiles at different time steps once the temperature in a semi-infinite body is suddenly increased from $u_0$ to $u_s$. The $x$ axis is aligned with the only direction along which temperature can vary.