2-D non-periodic homogenization of the elastic wave equation: 
\(SH\) case

Laurent Guillot,\(^1\) Yann Capdeville\(^1\) and Jean-Jacques Marigo\(^2\)

\(^1\)Équipe de sismologie, Institut de Physique du Globe de Paris, CNRS, France. E-mail: capdevi@ipgp.jussieu.fr
\(^2\)Laboratoire de Mécanique des Solides (UMR 7649), École Polytechnique, Palaiseau, France

SUMMARY
In the Earth, seismic waves propagate through 3-D heterogeneities characterized by a large variety of scales, some of them much smaller than their minimum wavelength. The costs of computing the wavefield in such media using purely numerical methods, are very high. To lower them, and also to obtain a better geodynamical interpretation of tomographic images, we aim at calculating appropriate effective properties of heterogeneous and discontinuous media, by deriving convenient upscaling rules for the material properties and for the wave equation.

To progress towards this goal we extend our successful work from 1-D to 2-D. We first apply the so-called homogenization method (based on a two-scale asymptotic expansion of the field variables) to model antiplane wave propagation in 2-D periodic media. These latter are characterized by short-scale variations of elastic properties, compared to the smallest wavelength of the wavefield. Seismograms are obtained using the 0th-order term of this asymptotic expansion, plus a partial first-order correction. Away from boundaries, they are in excellent agreement with solutions calculated at a much higher computational cost, using spectral elements simulations in the reference media. We then extend the homogenization of the wave equation, to 2-D non-periodic, deterministic media.

Key words: Numerical solutions; Seismic anisotropy; Computational seismology; Wave propagation.

1 INTRODUCTION
Are some parts of the Earth elastically anisotropic at the large scale? Even though some have performed sophisticated statistical analyses to prove that anisotropy is needed to explain data (e.g. the \(F\)-test in Trampert & Woodhouse 2002), this is still an open question: the answer given by seismological data, often depends on the parameterization of the media that elastic wavefields have propagated through. Classically, synthetic data calculated in isotropic models with a fine spatial parameterization of elastic properties, fit real data as well as synthetic ones, calculated using anisotropic models with a coarser parameterization.

Nevertheless, seismological imaging often relies on relatively long-period data (e.g. Beucler & Montagner 2006) with respect to the size of the Earth heterogeneities, as can be found in the crust or even in the upper mantle (Gudmundsson et al. 1990). And it is well known that in that case, long-wavelength wavefields naturally ‘upscale’ these media: what waves see, is a medium with effective properties (see Kennett 1983; Chapman 2004 for instance). It is then of prime interest, to be able to understand the rules of this upscaling process, or, in other words, to understand how media containing small-scale heterogeneities are ‘seen’ by the long-period part of an elastic wavefield.

Backus (1962) showed that finely layered isotropic media are orthotropic for long-wave propagation. Recently, Fichtner & Igel (2008) studied the effect of smoothing the Earth’s crust in numerical simulations, and conclusively showed that Backus’ result is valid even when the scales of heterogeneities and of the wavelength, are comparable. One of the important achievements of this paper (and of the companion paper, see Capdeville et al. 2010b) will be to show that this kind of result can be generalized to more complex media, like 2-D randomly generated ones. We shall therefore bring another interpretation to seismological data: perhaps the ‘real’ Earth may be widely isotropic but highly heterogeneous, and the ‘effective’ Earth, anisotropic.

As recalled by Capdeville et al. (2010b) in their introduction, numerous different kinds of methods have been used to determine the effective properties of media with small-scale heterogeneities. In this paper, we will tackle this issue using some tools of the so-called homogenization theory. Many studies have been devoted to this theory, either to its mathematical bases (Murat & Tartar 1985; Allaire 1992) or to some physical applications, often related to properties of composite materials (Dumontet 1986; Francfort & Murat 1986). Unfortunately, most of these works were related to periodic media, in the static case—remote from the preoccupations of a seismologist who study using wave properties, an Earth far from
being a simple checkerboard. Some works, however, were devoted to time-dependent equations (e.g. Sanchez-Palencia 1980; Auriault & Bonnet 1985; Fish et al. 2002; Fish & Chen 2004; Parnell & Abrahams 2006; Lurie 2009; Allaire et al. 2009) or to non-periodic settings: Briane (1994), who developed for elliptic equations, a theory based on a local, periodic approximation to non-periodic materials; Shkoller (1997), who applied Briane’s theory to defects in fibre-reinforced composites; or more recently, Nguetseng (2003), who presented a quite complex mathematical treatment of elliptic equations in non-periodic media.

High-order homogenization in the dynamical, non-periodic case, has been tackled by Capdeville & Marigo (2007, 2008) for wave propagation in stratified media. More recently, Capdeville et al. (2010a) proposed another method to understand how to upscale the wave equation, in 1-D; in that paper, they suggested a homogenization procedure that can be generalized to a higher dimension in space. In this paper, we will apply this procedure to 2-D SH-wave propagation (or antiplane elastodynamic motion). In a first part, we will recall some general features of homogenization theory applied to 2-D periodic media (as was done by Fish & Chen (2004) in the multidimensional case); and then we will generalize these results to the non-periodic case, using the heuristic procedure suggested in Capdeville et al. (2010a) or in the companion paper (Capdeville et al. 2010b). We will show that applying order 0 homogenization theory to the antiplane elastodynamic equation leads to results in very good agreement with solutions considered as reference ones (using the Spectral Element Method in very heterogeneous media; see e.g. Komatitsch & Vilotte 1998).

2 PERIODIC CASE

2.1 Generalities

Let us consider an infinite elastic plane (or a large 2-D finite domain surrounded by an absorbing boundary layer) whose elastic tensor and density are \( \mathbf{\epsilon}^0 \) and \( \rho^0 \), respectively. As in the 1-D case (Capdeville et al. 2010a), we consider this elastic plate to be infinite (or surrounded by an absorbing boundary layer, which is equivalent) to avoid the necessary treatment of boundary conditions. This issue is postponed to a future work.

The position of a point in this plane, relatively to a Cartesian basis defined by some basis vectors \( \mathbf{e}_1, \mathbf{e}_2 \), is described by a vector \( \mathbf{x} = (x_1, x_2) \), where \( \mathbf{e} \) is the transposition operator.

The propagation of elastic waves in such a medium, results from the application of an external force \( f(x,t) \), that will be defined later (note that this quantity is a scalar in the case of antiplane wave propagation, so \( f = f \)). The general, classical, linearized equation of motion and constitutive relation we have to solve for 2-D wave propagation in an elastic medium are

\[
\begin{align*}
\rho^0 & \partial_t \mathbf{u} - \nabla \cdot \mathbf{\sigma} = f, \\
\mathbf{\sigma} &= \mathbf{\epsilon}^0 \cdot \nabla \mathbf{u},
\end{align*}
\]

(1)

where \( \mathbf{u} \) is the displacement, \( \mathbf{\sigma} \) the incremental Cauchy stress tensor, \( \mathbf{\epsilon}^0 \) the general stiffness tensor, and \( \nabla \mathbf{u} = (\partial_{x_1} \mathbf{u}, \partial_{x_2} \mathbf{u}) \) the space gradient operator.

As we consider antiplane motion in 2-D, the displacement generated by the application of a (scalar) source term, is a scalar quantity, which will be noted \( u \). This displacement is perpendicular to the \((x_1, x_2) - \) plane. Eq. (1) can therefore be recast into these simplified expressions (Appendix A)

\[
\begin{align*}
\rho^0 \partial_t u - \nabla \cdot \mathbf{\sigma} &= f, \\
\mathbf{\sigma} &= \mathbf{\mu}^0 \cdot \nabla u,
\end{align*}
\]

(2)

where the elastic tensor \( \mathbf{\epsilon}^0 \) reduces to

\[
\mathbf{\mu}^0 = \begin{pmatrix} \mu_{11}^0 & \mu_{12}^0 \\ \mu_{21}^0 & \mu_{22}^0 \end{pmatrix}
\]

(3)

defined by only three independent parameters, as \( \mu_{12}^0 = \mu_{21}^0 \). Note that in the following, we shall not use the strain tensor in the definition of the constitutive law, as done in Capdeville et al. (2010b).

This will imply a slightly different procedure to retrieve the so-called homogenized elastic tensor, for we shall impose its symmetry.

The index notation will be extensively used throughout this paper.

Any tensor \( \mathbf{A} \) of order \( p \) has components, in the index notation: \( A_{i_1 \ldots i_p} \).

2.2 Set up of the homogenization problem—periodic case

Let us assume in this first section, that the physical properties of the surface, are periodic, that is two real numbers \( \ell_1 \) and \( \ell_2 \) can be found such that for any \( x, \rho^0(x + \ell_1 \mathbf{e}_1) = \rho^0(x + \ell_2 \mathbf{e}_2) = \rho^0(x) \) and \( \mathbf{\mu}^0(x + \ell_1 \mathbf{e}_1) = \mathbf{\mu}^0(x + \ell_2 \mathbf{e}_2) = \mathbf{\mu}^0(x) \) (Fig. 1). The medium is going to be said \((\ell_1, \ell_2)\)-periodic.

A characteristic length \( \ell \ (\ell \sim (\ell_1^2 + \ell_2^2)^{1/2}) \) for instance) can be introduced for such a heterogeneity pattern.

Let us also consider that the time dependence of the source term \( f \) (eq. 2) is characterized by a corner frequency \( f_c \) in the spectral domain. This hypothesis underlies the existence of a minimum wavelength \( \lambda_m \) for the propagating wavefield, away from the source.

Figure 1. Sketching detail of one (very simple) periodic reference cell of an infinite plane. The periodicity \((\ell_1, \ell_2)\) has a characteristic length \( \ell \) (see main text for a definition) much smaller than \( \Lambda \), the characteristic size of the propagating wavefield. Note that the heterogeneity pattern in the reference cell may be much more complex than in this figure, as long as it is periodic.
In this section dedicated to antiplane wave propagation in periodic settings, we will assume that
\[ \epsilon = \frac{\ell}{\lambda_m} \ll 1. \] (4)
Heterogeneities in the medium are therefore considered to be much smaller than the minimum wavelength of the propagating wavefield—or, in other words, the scale of heterogeneities is small, in comparison with the scale of the wavefield.

For \( \lambda_m \) fixed, we can define a sequence of wave propagation problems, by varying the initial periodicity (\( \ell_1, \ell_2 \)), each problem being then indexed by a specific \( \epsilon \). This approach is classical in the homogenization theory, the goal being then to find an asymptotic solution (\( u, \sigma \)) to the wave equation when \( \epsilon \) tends towards 0 (Sanchez-Palencia 1980). Note that our original problem is met only for the unique pair (\( \ell_1, \ell_2 \)), and therefore, a unique \( \epsilon \).

We rewrite eq. (2) indexing each physical quantity by \( \epsilon \), at the exception of the source (which is in fact not exactly correct for a point source, as underlined in Capdeville et al. (2010a), for instance):
\[ \rho^\epsilon \partial_{\epsilon t} u^\epsilon - \nabla \cdot \sigma^\epsilon = f^\epsilon, \]
\[ \sigma^\epsilon = \mu^\epsilon \cdot \nabla u^\epsilon. \] (5)

Our purpose is to study long-wavelength wave propagation with respect to the scale of heterogeneities. As underlined in “Introduction”, simple physical considerations (see, for instance, Bakhuis 1962; Chapman 2004) lead us to think that a wavefield “sees” small heterogeneities in an effective way at a large scale; but we can expect the wavefield should also be locally sensitive to rapid variations of elastic properties. A convenient trick to explicitly take small-scale heterogeneities into account, is to re-scale them, in order for them to artificially have the same scale as the wavefield. A space variable which varies rapidly is then introduced
\[ y = \frac{y}{\epsilon}. \] (6)
The variable \( y \) is usually called the microscopic variable and \( x \) is the macroscopic one. It can be seen that when \( \epsilon \to 0 \), any change in \( y \) induces a very small change in \( x \). To mathematically take both of these scales into account, we will assume that all physical quantities depend on both \( x \) and \( y \); a direct consequence of this, is the redefinition of the spatial gradient operator:
\[ \nabla_x \to \nabla_x + \frac{1}{\epsilon} \nabla_y. \] (7)
This introduces powers of \( \epsilon \) in the equation of motion and constitutive relation. The homogenization theory is a tool that allows to catch both macro- and microscopic effects, by then seeking solutions to the wave eq. (5) under the form of asymptotic expansions in powers of \( \epsilon \) (note that only powers of \( \epsilon \) will appear in the following, so no ambiguity should arise with superscripts of \( u \) and \( \sigma \))
\[ u^i(x, t) = \sum_{i=0}^{\infty} \epsilon^i u^i(x, x/\epsilon, t) = \sum_{i=0}^{\infty} \epsilon^i u^i(x, y, t), \]
\[ \sigma^i(x, t) = \sum_{i=1}^{\infty} \epsilon^i \sigma^i(x, x/\epsilon, t) = \sum_{i=1}^{\infty} \epsilon^i \sigma^i(x, y, t). \] (8)
In these equations, \( u^i \) and \( \sigma^i \) depend on both space variables \( x \) and \( y \), and are chosen to be (\( \ell_1/\epsilon, \ell_2/\epsilon \))-periodic in \( y \). Note that the expansion for the stress starts at \( i = -1 \) because of the constitutive relation between \( u \) and \( \sigma \), and the redefinition of the gradient operator in eq. (7).

The \( y \)-dependence of material properties is defined as
\[ \rho(y) = \rho^\epsilon(\epsilon y), \]
\[ \mu(y) = \mu^\epsilon(\epsilon y). \] (9)
These quantities are called, the reference cell density and stiffness tensor. They are of course, \( \epsilon \)-independent and \( (\ell_1/\epsilon, \ell_2/\epsilon) \)-periodic.

The external source term \( f \) is assumed to be \( \epsilon \) and \( y \) independent. This assumption is not obvious for a point source. The idea behind this assumption is to forget about the source during the asymptotic development and then to reintroduce it using energy considerations. This issue is discussed with more details in Capdeville et al. (2010a) or Capdeville et al. (2010b).

Finally, we define the so-called cell average (over the reference cell \( y = [0, \ell_1/\epsilon] \times [0, \ell_2/\epsilon] \), \( y \) being its surface), for any (tens)orial function \( \mathbf{g}(x, y) \) which is \( (\ell_1/\epsilon, \ell_2/\epsilon) \)-periodic in \( y \)
\[ \langle \mathbf{g} \rangle(x) = \frac{1}{|y|} \int_{y} \mathbf{g}(x, y)dy. \] (10)
For any tensorial quantity \( \mathbf{g}(x, y) \) of order \( q \) and \( (\ell_1/\epsilon, \ell_2/\epsilon) \)-periodic in \( y \), it can be shown that
\[ \langle \partial_y \mathbf{g}(\ell_1/\epsilon, \ell_2/\epsilon) \rangle = 0. \] (11)
For any couple of tensorial functions \( \mathbf{g}(x, y) \) and \( \mathbf{h}(x, y) \), of order \( p \) and \( q \) respectively, and \( (\ell_1/\epsilon, \ell_2/\epsilon) \)-periodic in \( y \), is verified the following property
\[ \langle \partial_y \mathbf{g}(\ell_1/\epsilon, \ell_2/\epsilon) \partial_y \mathbf{h}(\ell_1/\epsilon, \ell_2/\epsilon) \rangle = 0, \] (12)
and then
\[ \langle \partial_y \mathbf{g}(\ell_1/\epsilon, \ell_2/\epsilon) \partial_y \mathbf{h}(\ell_1/\epsilon, \ell_2/\epsilon) \rangle = - \langle \mathbf{g}(\ell_1/\epsilon, \ell_2/\epsilon) \partial_y \mathbf{h}(\ell_1/\epsilon, \ell_2/\epsilon) \rangle. \] (13)
Let us now turn to the iterative resolution of our homogenization problem.

### 2.3 Resolution of the two-scale homogenization problem

In the following, the time dependence \( t \) is dropped to ease the notations.

Introducing expansions (8) into eq. (5), using (7) and identifying term by term in \( \epsilon^i \), we obtain
\[ \rho \partial_{\epsilon t} u^i - \nabla \cdot \sigma^i - \nabla \cdot \sigma^{i+1} = f \delta_{i,0}, \] (14)
\[ \sigma^i = \mu \cdot (\nabla u^i + \nabla u^{i+1}). \] (15)
These last equations have to be solved for each \( i \).

(i) Eq. (14) for \( i = -2 \) and eq. (15) for \( i = -1 \) give
\[ \nabla \cdot \sigma^{-1} = 0, \]
\[ \sigma^{-1} = \mu \cdot (\nabla u^0). \] (16)
These equations imply that
\[ \nabla \cdot (\mu \cdot (\nabla u^0)) = \partial_y (\mu \partial_y u^0) = 0. \] (17)
Multiplying the last equation by \( u^0 \), integrating over the reference cell, and then by part, taking the periodicity of \( u^0 \) and \( \mu \) into account, leads to
\[ \int_{\Omega} \mu \partial_y u^0 \partial_y u^0 dy = 0. \] (18)
As \( \mu \) is positive-definite, the unique solution to the above equation is
\[ u^0(x, y) = u^R(x). \] We therefore have
\[ \sigma^{-1} = 0, \]
\[ u^0 = u^R(x) = [u^0]. \] (20)
This last equality implies that the order 0 solution in displacement is independent on the fast variable \( y \), and therefore does not oscillate with \( x/\varepsilon \). This is a major result that confirms the well-known fact that the displacement field is poorly sensitive to scales much smaller than its own scale (see Chapman 2004). Anecdotally, this fact justifies the name of the theory: \( u^0 \) is an homogenized solution.

(ii) Eq. (14) for \( i = -1 \) and eq. (15) for \( i = 0 \) give

\[
\nabla_y \cdot \sigma^0 = 0 ,
\]

(21)

\[
\sigma^0 = \mu \cdot (\nabla_y u^1 + \nabla_x u^0) ,
\]

(22)

or rewritten using the summation convention for repeated indices

\[
\sigma^0 = \mu_{ij}(\partial_{ij} u^1 + \partial_{ij} u^0) .
\]

(23)

Because of the action of the divergence operator, eq. (21) does not imply that \( \sigma^0(x, y) = \sigma^0(x) = (\sigma^0)^*(x) \), as it the case in 1-D (Capdeville et al. 2010a), where the divergence merely is the gradient operator. Nevertheless, we will next obtain a simple expression for \( (\sigma^0)^* \).

Both eqs (21) and (22) lead to this next equality

\[
\nabla_y \cdot (\mu \cdot (\nabla_y u^1 - \nabla_y - (\mu \cdot \nabla_x u^0)) - \nabla_y \cdot (\mu \cdot \nabla_x u^0) ,
\]

(25)

which should be verified, whatever the gradient of \( u^0 \) be. Because of this and of the linearity of the previous equation, we can look for a solution under the form

\[
u^1(x, y) = \langle u^1 \rangle(x) + \chi^1(x) \cdot \nabla_x u^0(x).\]

(26)

The vectorial quantity \( \chi^1(x) \) (with components \( \chi^1_i(x), \quad k = 1, 2 \), is called the first-order periodic corrector; it is \((\ell_1/\varepsilon, \ell_2/\varepsilon)\)-periodic. Introducing (26) into (25), we obtain the equations of the so-called cell problems

\[
\nabla_y \cdot (\mu \cdot (\nabla_y + \nabla_y \chi^1)) = \partial_{ii} \left( \mu_{ij} \left( \delta_{ik} + \partial_{ik} \chi^1_i \right) \right) = 0 ,
\]

(27)

that lead to find the first-order corrector components. Given (26), are verified \( \langle \chi^1 \rangle = 0 \). Is the identity tensor.

On the contrary to the 1-D case where an analytical solution to the previous equation does exist, we can not solve it in 2-D, but numerically. The special case of a stratified (and non-periodic) model, which can be considered as a 1-D medium, is treated in Appendix C.

In prevision to the next section, where homogenization in non-periodic settings will be tackled, let us rewrite (27) under the equivalent form

\[
\nabla_y \cdot H = \nabla_y \cdot (\mu \cdot G) = 0
\]

(28)

where \( H = G = 1 + \nabla_y \chi^1 \) are second-order tensors whose components have the dimension of a stress and of the gradient of a displacement, respectively. This observation will be useful in the section dedicated to non-periodic homogenization.

Taking the cell average of (24) and using the ansatz (26), we obtain the order 0 constitutive relation

\[
\langle \sigma^0 \rangle = \mu^* \cdot \nabla_x u^0 ,
\]

(29)

where \( \mu^* \) is the order 0 homogenized elastic tensor whose components are

\[
\mu^*_{ij} = \langle \mu_{ik} (\delta_{jk} + \partial_{jk} \chi^1_k) \rangle .
\]

(30)

Because of (22), (29) and the definition of \( G \), we obtain

\[
\mu^* = \langle H \rangle .
\]

(31)

At the first sight, it is not obvious to physically interpret the expression for the effective elastic constants (30). As noticed by Papanicolaou & Varadhan (1979), they are the sum of their own average and of a correction term, which is the average of elementary stresses associated to displacements equal to the first-order corrector’s components. This observation may lead to a practical method for the determination of effective elastic constants, as shown by Suquet (1982) in the static case, and Grechka (2003) in the dynamical one. We will use this kind of observation in the next section (dedicated to the non-periodic case), to determine conveniently, both the \( y \)-dependence of the stiffness tensor, and the effective, homogenized tensor associated to it.

(iii) Eq. (14) for \( i = 0 \) and eq. (15) for \( i = 1 \) give

\[
\rho \partial_{tt} u^0 - \nabla_y \cdot \sigma^0 - \nabla_y \cdot \sigma^1 = f ,
\]

(32)

\[
\sigma^1 = \mu \cdot (\nabla_x u^2 + \nabla_x u^1) .
\]

(33)

Applying the cell average on (32), using the property (11), the fact that \( u^0 \) does not depend on \( y \), and the expression (29) for the average of \( (\sigma^0) \), we obtain the order 0 wave equation

\[
\rho^* \partial_{tt} u^0 - \nabla_y \cdot \langle \sigma^0 \rangle = \langle f \rangle = f ,
\]

(34)

\[
\langle \sigma^0 \rangle = \mu^* \cdot \nabla_x u^0 ,
\]

where \( \rho^* = \langle \rho \rangle \) is the effective density, whose expression is the same as in 1-D (Capdeville et al. 2010a). We can notice that this equation only is the classical wave equation that can be solved using classical techniques. Because \( \rho^* \) and \( \mu^* \) are constant, solving the order 0 wave equation is a much simpler task than in the original medium; no numerical difficulty arises, that is related to the rapid spatial variations of the physical properties in the plane.

Once \( u^0 \) is found, and because the cell problem (27) has been solved for \( \chi^1 \) to determine the effective elastic tensor, the first-order correction \( \chi^1(x/\varepsilon) \cdot \nabla_x u^0(x) \) can then be computed.

Obtaining the complete order 1 solution \( u^1(x, y) \) using (26) (i.e., finding the remaining term \( \langle u^1(x) \rangle \) is more complex than in 1-D, as we shall show now.

Subtracting (34) to (32) we obtain

\[
\nabla_y \cdot \sigma^1 = (\rho - \langle \rho \rangle ) \partial_{tt} u^0 ,
\]

(35)

which, together with (33), gives

\[
\nabla_y \cdot (\mu \cdot \nabla_x u^2) = - \nabla_y \cdot (\mu \cdot \nabla_x u^1) + (\rho - \langle \rho \rangle ) \partial_{tt} u^0 .
\]

(36)

Using (26), we then obtain the following, expanded equation

\[
\partial_{ii} (\mu_{ij} \partial_{ij} u^2) = - \partial_{ii} (\mu_{ij} \partial_{ij} u^1) - \partial_{ii} (\mu_{ij} \partial_{ij} u^0) + (\rho - \langle \rho \rangle ) \partial_{tt} u^0 .
\]

(37)

Exactly as it was done in (26), using the linearity of the last equation, we can separate variables and look for a solution under the following form:

\[
u^1(x, y) = \langle u^2 \rangle(x) + \chi^2(x) \rho_{ij}(\delta_{ij} + \partial_{ij} \chi^2_i) + \chi^2_{ii}(x) \rho_{ij} \partial_{ij} u^0(x) .
\]

(38)

The components of the second-order tensor \( \chi^2 \), and \( \chi^2_{ii} \), are solutions of the following partial differential equations:

\[
\partial_{ii} (\mu_{ij} (\delta_{ij} \chi^2_i + \partial_{ij} \chi^2_i)) = 0 .
\]

(39)

\[
\partial_{ii} (\mu_{ij} \partial_{ij} \chi^2_{ii}) = \rho - \langle \rho \rangle ,
\]

(40)

where \( \chi^2_i \) and \( \chi^2_{ii} \) are \((\ell_1/\varepsilon, \ell_2/\varepsilon)\)-periodic. To ensure the uniqueness of the solutions, we impose for any \( (l, k) \), \( \langle \chi^2_{kl} \rangle = \langle \chi^2_{ii} \rangle = 0 .
\]

Note that \( \chi^2 \) is symmetric: \( \chi^2_{ij} = \chi^2_{ji} \).
Introducing (38) into (33) and taking the cell average over the cell, we obtain the order 1 contribution to the constitutive relation
\[ \langle \sigma^1 \rangle = \mu^* \partial_{ij} \langle u^1 \rangle + \mu^* \partial_{ij} \delta_{ik} \langle u^0 \rangle + \mu^* \partial_{ij} \langle u^0 \rangle, \] (41)
where
\[ \mu^*_{ij} = \langle \mu_{ij} (\delta_{ik} \chi^0_{ik} + \partial_{ik} \chi^0_{ik}) \rangle \] and
\[ \mu^*_{ik} = \langle \mu_{ik} (\partial_{ik} \chi^0_{ik}) \rangle. \] (43)

Finally, taking the cell average of eq. (14) for \( i = 1 \) and using eq. (41), gives the order 1 contribution to the wave equation:
\[ \langle \rho \rangle \partial_{tt} \langle u^1 \rangle + \langle \rho \chi^0 \rangle \partial_{xj} \partial_{xj} \langle u^1 \rangle - \partial_{xj} \langle \sigma^1 \rangle = 0, \] (44)
\[ \langle \sigma^1 \rangle = \mu^* \partial_{ij} \langle u^1 \rangle + \mu^* \partial_{ij} \delta_{ik} \langle u^0 \rangle + \mu^* \partial_{ij} \langle u^0 \rangle. \] (45)

Unfortunately, the \( \mu^*_{ij} \) coefficients are not null in general, therefore we cannot further simplify the order 1 equations, to a form similar to that obtained for 1-D wave propagation. However, as noted by Boutin (1996)—and this can be verified numerically—these coefficients are quite small in comparison to the order 0 homogenized coefficients \( \mu^0_{ij} \), and the associated term could be considered as negligible. Furthermore, they are identically null when homogenized elastic properties in the reference cell, are isotropic (Boutin 1996); this is the special case of a heterogeneous but macroscopically isotropic material.

Though we may solve up to a higher order (see, for instance, Fish & Chen 2004, for a 2-D periodic case), we stop here the expansion.

2.4 Practical resolution

Solving the homogenized wave equations derived above can be done in a series, or by combining successive orders together, using one of these kinds of wave propagation solvers: normal-mode summation (Capdeville & Marigo 2007); finite element methods (Fish & Chen 2004); or, as in Capdeville et al. (2010a) and this work, the spectral element method (SEM, see for instance Komatitsch & Vilotte 1998).

In the most general case, we want to solve
\[ \langle \vec{u}^{<1>} (x) \rangle = u^1(x) + \epsilon \langle u^1 \rangle (x) + \cdots + \epsilon^\rho \langle u^\rho \rangle (x), \] (46)
\[ \langle \vec{\sigma}^{<1>} (x) \rangle = \langle \sigma^0 \rangle (x) + \epsilon \langle \sigma^1 \rangle (x) + \cdots + \epsilon^\rho \langle \sigma^\rho \rangle (x), \] (47)
where the bracketed terms are the different homogenized fields, that we could calculate as in the previous section. Once these effective fields known, we could find the complete ones, \( \vec{u}^{<1>} \) and \( \vec{\sigma}^{<1>} \), by applying a high-order corrector operator to (47), as suggested by Capdeville et al. (2010a) in 1-D. Then it can be shown that the following approximations are verified:
\[ \vec{u}^{<1>} = u^1 + O(\epsilon^\rho), \]
\[ \vec{\sigma}^{<1>} = \sigma^1 + O(\epsilon^\rho). \] (48)

As noticed previously, the order 1 effective equation of motion and constitutive relation, cannot be reduced to the classical elastic wave equations, unless \( \mu^* \) is the null tensor; this implies that in the most general case, the obtention of \( \langle u^1 \rangle (x) \) and \( \langle \sigma^1 \rangle (x) \) involves important modifications in the SEM code. Of course we could proceed to homogenization to order 1, starting from a microscopically (at the scale of the reference cell) isotropic medium, and obtaining a partial order 2 solution. Nevertheless, we have not lead this work, the case of an isotropic and periodic medium, being irrelevant in geophysics.

As we shall see in the Section 2.6 dedicated to some examples of wave propagation in a periodic setting, determining only a partial first-order solution seems enough to obtain quite accurate seismograms, with respect to the ones calculated in the reference heterogeneous medium. Therefore, as in Capdeville et al. (2010b), we will only try to solve
\[ \vec{u}^{<1/2>} (x) = \langle \vec{u}^{<0>} \rangle (x) + \chi^1 (x/\epsilon) \cdot \nabla_x \langle \vec{u}^{<0>} \rangle (x), \] (49)
where the 1/2 superscript means ‘partial order 1’. Of course, it is only a partial order 1 solution, and in general
\[ u^1 (x) \neq \vec{u}^{<1/2>} (x) + O(\epsilon^2), \] (50)
on the contrary to the 1-D case (Capdeville et al. 2010a), unless \( \mu^1 \) is very small (or null).

2.5 External source term

In seismology, we generally consider that the source dimension is much smaller than the smallest wavelength of the (far) wavefield, and that a point source (located around a given \( x_0 \)) is therefore a good approximation. This localization of the source leads to its mathematical formulation: \( f(x, t) = \delta(x - x_0)g(t) \).

We claimed earlier in this paper, that the external force does not depend on the \( \epsilon \)-parameter. As shown in Capdeville et al. (2010a) this is not true in general, because interactions of a (ideal) point source with its surrounding microscopic structure must be accounted for.

In this paper, the corrective source term of order 1 will not be needed, as the asymptotic homogenized solution we look for will be expanded at its sole, truncated order 1, and that only point forces located in homogeneous source regions will be considered. Nevertheless the reader should keep in mind that this is an oversimplification, and that the procedure suggested in Capdeville et al. (2010a) or Capdeville et al. (2010b)—and based on energetic considerations—is necessary to treat source effects in a correct manner.

2.6 Homogenization in a periodic setting: an example

Let us turn now to a practical application of the homogenization theory in a periodic setting. A numerical experiment is led in a rectangular plane (Fig. 2) of size \( 15 \times 20 \) km², surrounded by absorbing boundary layers. The microscopic structure is that of a stretched checkerboard, as shown in Fig. 1, with a horizontal periodicity of 120 m, and a vertical one of 200 m. Elastic properties and density in the elements of the periodic cell, are ±50 per cent around a mean value of 60 GPa for \( \mu_{11} \) and \( \mu_{22}(\mu_{12} \text{ being null) and 2800 kg m}^{-3} \text{ for } \rho \).

To obtain values for the first-order corrector components and their spatial derivatives, the cell problem (27) is solved in the reference cell using periodic boundary conditions. The differential equations are solved with a finite element method based on the same mesh and quadrature than the one that will be used to solve the wave equation. This then allows to compute the homogenized stiffness tensor \( \mu^* \).
2-D non-periodic homogenization, SH case

and density $\rho^*$, and the 0th-order homogenized wave equation,

$$\langle \rho \rangle \partial_{tt} u - \nabla_x \cdot (\mu^* \cdot \nabla_x u) = f,$$

(51)

using the spectral element method.

A point source is located at $S = (x_S^1, x_S^2) = (5 \text{ km}, 10 \text{ km})$. Its time evolution is described by a Ricker with a central time shift of 0.4 s and a central frequency of 5 Hz (to which is associated a corner frequency of about 12.5 Hz). A receiver is located at $R = (x_R^1, x_R^2) = (10 \text{ km}, 10 \text{ km})$, therefore, at the same vertical position as the source, which means, that the periodicity seen at this receiver point, is of 120 m. Considering the physical properties of the medium (with a minimum wavelength of around 370 m), and the cut-off frequency of the source, the value of $\varepsilon$ in the far field, is around 0.32. To properly compare the homogeneous solution with a reference one calculated in the reference checker boarding box, both solutions are computed with the same mesh (each element of this latter, corresponding to an element of the checkerboard), and the same time step ($10^{-3}$ s).

The seismograms resulting from these numerical simulations are reported in Fig. 3, in which are shown the reference solution (grey line), the order 0 homogenized solution (black line) and a solution obtained when taking the arithmetic average of both density and elastic constants (dashed line). This last solution (which could be seen as a ‘natural averaging’ solution) is not in phase with the reference one, whereas there is an excellent agreement between this latter and the order 0 homogenized seismogram.

Snapshots along a line (L) (Fig. 2) between the source and the receiver, are shown in Fig. 4(a) for each of the simulations, at $t = 0.4$ s. Once again, the agreement is excellent between reference and 0th-order homogenized solutions, and very poor between the ‘natural filtering’ solution and the reference one.

The residual between the reference solution and the order 0 homogenized solution $u^0(x, t) - \hat{u}^0(x, t)$, at $t = 0.4$ s, is plotted in Fig. 4(b). The error amplitude is lower than the percent and contains fast variations.

Once done the 0th-order homogenized simulation (and therefore, once known displacement gradients), the incomplete order 1 solution (49) can be computed.

In Fig. 4(c) is shown the partial order 1 residual $u^1(x, t) - \hat{u}^{1/2}(x, t)$. Comparing Figs 4(b) and (c), it appears that the partial order 1 periodic correction removes most of the fast variations in the order 0 residual. The remaining fast variations are due to the neglected higher-order terms, in the expansion (48). The smooth remaining residual could be due to the $\langle u^2 \rangle$ term (and also, to

© 2010 The Authors, *GJI*, 182, 1438–1454
Journal compilation © 2010 RAS
Figure 3. Displacement (in meters) recorded at the receiver $R$ located at (10 km, 10 km). The reference is plotted in grey, the order 0 homogenized solution in black and the ‘natural averaging’ (see text) solution is in dashed line.

Figure 4. Snapshots along the (L)-line (Fig. 2). (a) Grey line: snapshot of the displacement $u^r(x, t)$ at $t = 0.4$ s computed in the reference model. Black line: the order 0 homogenized solution $\hat{u}^0(x, t)$. Dashed line: solution computed in a model obtained by arithmetically averaging the elastic properties. (b) Order 0 residual, $u^r(x, t) - \hat{u}^0(x, t)$. (c) Partial order 1 residual, $u^r(x, t) - \hat{u}^{1/2,ε}(x, t)$. (d) Grey line: partial order 1 residual for $ε = 0.32$. Black line: partial order 1 residual for $ε = 0.16$ with an amplitude multiplied by 4, at $t = 1.2$ s.
higher-order ones in 47) that is (are) not computed. To check the order of that this smooth remaining residual, we perform a test similar (at \( t = 1.2 \) s) to the previous one, but with \( \varepsilon = 0.16 \) (which corresponds to a periodicity divided by a factor of 2). Surprisingly, there is approximately a factor of 4 in amplitude between both, which indicates a residual which decreases as \( \varepsilon^2 \), almost (Fig. 4d). That means that the \( \langle u^1 \rangle \) term is close to being null and that (45) may be approximately rearranged as in Capdeville et al. (2010a) \( \langle u^1 \rangle \) being equal to 0 in 1-D. Equivalently, that means that homogenized elastic properties in the reference cell are close to be isotropic, which is in fact the case for this example, and that the coefficients of \( \mu^{1*} \) are quite small in (45).

3 NON-PERIODIC CASE

Let us now turn to the case of interest in geophysics: the Earth material properties being not spatially arranged periodically, we give up this drastic hypothesis and consider more complex patterns for heterogeneities. More precisely: we do not make any assumption on the spatial variability of density and elastic coefficients. After a theoretical treatment, we will apply it to the specific case of a plane surface in which properties are randomly generated around a constant mean value, surrounded by a strip of progressively constant physical properties (to avoid any spurious reflection) and a perfectly matched layer (PML, e.g. Festa & Vilotte 2005) layer (to avoid the treatment of boundary layers in our homogenization procedure), as shown in Fig. 5(a). Note that the issue we shall tackle, is not that of the homogenization of random structures as studied by Papanicolaou & Varadhan (1979): in our example, the properties will be spatially known and unique.

We will still assume a minimum wavelength \( \lambda_m \) (or a maximum wavenumber \( k_m = 1/\lambda_m \)) for the antiplane wavefield \( u \), far enough from the source. Therefore, in some sense, heterogeneities in the plane, whose size is much smaller than \( \lambda_m \), should have a little, effective influence on the wavefield \( u \), and an homogenization procedure may be hopefully performed. As recalled by Aki & Richards (1980), there are three different wave propagation regimes (waves in a smoothly varying body, coda waves and the homogenized part of a wavefield) depending on the ratio of the wavefield characteristic scale, to the one of the heterogeneities.

But when properties are not periodic in space: what is the characteristic scale of the heterogeneities? The difficulty we are facing is therefore to find a clear spatial scale delimitation (to apply an homogenization procedure), to catch wavefield properties in each of these regimes. In 1-D, Capdeville & Marigo (2007) suggested to apply a filter to the recast elastic operator in the wavenumber domain, or to some physical quantities known a priori (Capdeville et al. 2010a); this filtering operation allowed them to obtain an accurate ‘homogenized’ solution, when compared to the reference one. Unfortunately, in a higher space dimension, we do not know to which physical quantities this filtering procedure has to be applied. We will suggest one way to proceed (as done in Capdeville et al. 2010a), after having recalled some filtering notions, and having set up a heuristic homogenization procedure to wave propagation in 2-D non-periodic domains.

3.1 Basic notions on spatial filtering

In the following theoretical development, we shall desire to separate low from high wavenumbers \( k = 1/\lambda \) (the norm of the wavenumber vector \( k \)) of a spatial distribution of any given tensorial quantity \( g(x) \), around a given wavenumber \( k_0 \). Note that the choice of \( k_0 \) will be determinant to accurately describe waveform properties in each wave propagation regime.

To that purpose we shall introduce a low-pass space filter operator that takes the form, for any function \( g \)

\[
F_{k_0}(g)(x) = \int_R g(x') w_{k_0}(x-x')dx',
\]

(52)

Figure 5. (a) Square, random model. The point source is located at S, the 14 receivers are also indicated. Here are represented the values of the \( \mu^{11}_0 \) coefficient. (b) Homogenized (or effective) model for the square, random model. In this figure are reported the values for the homogenized coefficient \( \mu^{11*}_e \), with \( \varepsilon_0 \) equal to 0.27.

© 2010 The Authors, GJI, 182, 1438–1454
Journal compilation © 2010 RAS
where \( w_{k_0} \) is a wavelet that would ideally be defined in the spectral domain as
\[
\tilde{w}_{k_0}(k) = \begin{cases} 1 & \text{if } k \leq k_0, \\ 0 & \text{if } k > k_0, \end{cases}
\]

(53)
where \( \tilde{w} \) is just the Fourier transform of \( w \).

In practice, to have a wavelet \( w_{k_0} \), for which a compact support is a good approximation, we do not use a filter with such a sharp cut-off but one defined by a smoother transition from 1 to 0 around \( k_0 \). There are many ways to design a filter with such a property. The one we shall use is characterized as
\[
\tilde{w}(k) = \begin{cases} 1 & \text{if } k \leq k_{\min}, \\ \frac{1}{2} \left[ 1 + \cos \left( \pi \frac{|k| - k_{\min}}{k_{\max} - k_{\min}} \right) \right] & \text{if } |k| \in [k_{\min}, k_{\max}], \\ 0 & \text{if } |k| \geq k_{\max}. \end{cases}
\]

(54)
where \( k_{\min} \) and \( k_{\max} \) are two real numbers around 1, defining a tapering zone from 1 to 0 for the low-pass filter. Just notice that the following property is verified: \( \int_{\mathbb{R}} w(x) dx = 1 \), and that we can easily define \( w_{k_0}(x) = k_0 w(x/k_0) \), the same but contracted (if \( k_0 > 1 \)) wavelet of corner spatial frequency \( k_0 \), that also verifies: \( \int_{\mathbb{R}} w_{k_0}(x) dx = 1 \). The choice of \( k_{\min} \) and \( k_{\max} \) is ad hoc and left to the user, this latter being aware of two limitations: the perfectly sharp cut-off:
\[
(k_{\min} = k_{\max} = 1)
\]
is characterized by an infinite support in the space domain, that cannot be truncated with a convenient accuracy; and any other choice, for which this truncation can be safely applied, does not present the interesting property of a perfectly clear separation of spatial scales for a given quantity \( g \).

Let us note finally, that for the ideal low-pass filter with a sharp cut-off:
\[
\mathcal{F}^0(h^0) = \mathcal{F}^0.
\]

(55)
This property is only approximated with a smooth transition around the cut-off.

3.2 Setup of the homogenization problem in the non-periodic case

In 1-D, Capdeville et al. (2010a) applied the previous filtering operation to physical quantities that were explicitly known \textit{a priori}, thanks to an analytical solution to the cell problem. For wave propagation in a higher space dimension, such an analytical solution does not exist, and do not know how to separate scales for density and elastic constants, to proceed to an homogenization of these quantities and of the wave equation. In other words, we do not know, for a given distribution of material properties, how to construct the \( x \) and \( y \) contributions of \( \rho \) and \( \mu \), from \( \rho^0 \) and \( \mu^0 \). To that purpose, we then present here, an original but heuristic procedure.

Classically, a small parameter \( \varepsilon \) is introduced to solve the two-scale homogenization problem
\[
\varepsilon = \frac{\lambda}{\lambda_m},
\]

(56)
where \( \lambda \) is a spatial wavelength, upon which will depend our asymptotic expansion. For a periodic medium, \( \lambda \) would be the local length defining the periodicity of the model (called \( \ell \) in the first section of this paper). In the non-periodic case, the introduction of another parameter is required
\[
\varepsilon_0 = \frac{\lambda_0}{\lambda_m},
\]

(57)
where \( \lambda_0 \) is the length below which a wavelength can be considered as belonging to the small-scale (microscopic) domain, and reciprocally for the large (or macroscopic) scale. This \( \lambda_0 \) parameter can be arbitrarily chosen; nevertheless, it makes sense to assume that the wavefield does interact with heterogeneities whose size is smaller than \( \lambda_0 \). Therefore, picking a \( \varepsilon_0 \ll 1 \), which means considering as microscopic, heterogeneities whose size is much smaller than the minimum wavelength, should be a good guess.

Exactly as done in the periodic case, we can now introduce a fast space variable \( y = x/\varepsilon \), consider \( x \) and \( y \) as macroscopic and microscopic space variables respectively, and redefine the gradient operator as in (7).

Then we introduce a wavelet \( w_{k_0}(y) = w_{k_0}(y) \) where \( w_{k_0} \) is a low-pass filter as defined in 3.1 and \( k_0 = 1/\lambda_m \). We assume that we can design it in such a way that its support in the space domain is contained in an interval \([-\alpha\lambda_m, +\alpha\lambda_m]^2 \), where \( \alpha \) is a positive real number.

Let \( y_0 = [-\beta\lambda_m, \beta\lambda_m]^2 \) be a square of \( \mathbb{R}^2 \), with \( \beta \) a positive real number larger than \( \alpha \), and \( y_0 \) the same square, but translated by a vector \( x/\varepsilon_0 \). We define \( T = [g(x, y) : \mathbb{R}^2 \times y_0 \to \mathbb{R}, y_0 \text{-periodic in } y] \) the set of functions defined in \( y \) on \( y_0 \) and periodically extended to \( \mathbb{R}^2 \). Associated to the wavelet \( w_{k_0} \), can be defined a filtering operator for any function \( g \in T \):
\[
\mathcal{F}(g)(x, y) = \int_{\mathbb{R}^2} g(x, y') w_{k_0}(y - y') dy'.
\]

(58)
This filter is a low-pass filter as defined in Section 3.1, for which the property (55) is verified.

Finally let \( V \) be the set of functions \( g(x, y) \) such that, for a given \( x \), the \( y \) part of \( g \) is periodic and contains only spatial frequencies higher than \( k_m \), plus a constant value in \( y \):
\[
V = \{ g \in T : \mathcal{F}(g)(x, y) = (g)(x) \},
\]

(59)
where, similarly to the periodic case, the cell average of \( g \) over the newly defined periodic cell is
\[
(g)(x) = \frac{1}{|y_0|} \int_{y_0} g(x, y) dy.
\]

(60)
Because the periodicity condition is kept in our procedure to treat wave propagation in non-periodic media, properties (11) and (13) are still valid. Furthermore, it can be shown that the (spatial) partial derivatives of the components of any (tensorial) function \( g \) in \( V \), are also in \( V \); and finally that
\[
\forall h \in V \text{ with } (h) = 0 \text{ and } \nabla_y g = h \Rightarrow g \text{ lies in } V.
\]

(61)
In this section and the next one, we proceed the same way as Capdeville et al. (2010a). We first assume that we have been able to define \( (\rho^{0,\varepsilon}(x, y), \mu^{0,\varepsilon}(x, y)) \) that verify
\[
(\rho^{0,\varepsilon}(x, y/\varepsilon_0) = \rho^0(x)
\]

(62)
and that set up a sequence (as in the periodic case) of models indexed by \( \varepsilon \)
\[
(\rho^{0,\varepsilon}(x) \equiv \rho^{0,\varepsilon}(x, \varepsilon_0), \\
(\mu^{0,\varepsilon}(x) \equiv \mu^{0,\varepsilon}(x, \varepsilon_0).
\]

(63)
We also assume that, with such a set of parameters, a solution to the homogenization problem described earlier exists. This is by far not obvious. The construction of such \( (\rho^{0,\varepsilon}(x, y) \text{ and } \mu^{0,\varepsilon}(x, y)) \) from \( (\rho^0(x) \text{ and } \mu^0(x)) \), which is a critical issue, is left for Section 3.4.
As in the periodic case (and after recasting of the wave equation, see Appendix A), we then look for the solutions of the wave equation and constitutive relation

$$\rho^{0,\varepsilon} \frac{\partial}{\partial t} u^{0,\varepsilon} - \nabla \cdot \sigma^{0,\varepsilon} = f,$$

$$\sigma^{0,\varepsilon} = \mu^{0,\varepsilon} \cdot \nabla u^{0,\varepsilon}.$$  \hfill (64)

A solution to the eq. (64) is again sought as an asymptotic expansion in powers of $\varepsilon$

$$u^{0,\varepsilon}(x, t) = \sum_{i=0}^{\infty} \varepsilon^i u^{i,\varepsilon}(x, x/\varepsilon, t) = \sum_{i=0}^{\infty} \varepsilon^i u^{i,\varepsilon}(x, y, t),$$

$$\sigma^{0,\varepsilon}(x, t) = \sum_{i=0}^{\infty} \varepsilon^i \sigma^{i,\varepsilon}(x, x/\varepsilon, t) = \sum_{i=0}^{\infty} \varepsilon^i \sigma^{i,\varepsilon}(x, y, t),$$

with the additional hypothesis—in this specific, non-periodic case—that $u^{0,\varepsilon}$ and $\sigma^{i,\varepsilon}$ must belong to the space $\mathcal{V}$. Introducing the expansions (65) in the wave equation (64) and using (7) we obtain the system of differential equations

$$\rho^{0,\varepsilon} \frac{\partial}{\partial t} u^{i,\varepsilon} - \nabla_y \cdot \sigma^{i,\varepsilon} - \nabla_y \cdot \sigma^{i+1,\varepsilon} = f \delta_{i,0},$$

$$\sigma^{0,\varepsilon}(x, t) = \mu^{0,\varepsilon} \cdot (\nabla u^{0,\varepsilon} + \nabla u^{0,\varepsilon+1}),$$  \hfill (66)

(67)

which need to be solved for each $i$, up to a given $i_0$. As in the periodic case (Section 2.4), we will restrict this resolution to order 1/2, that means: $i_0 = 0$, plus the first-order periodic correction.

### 3.3 Resolution of the two-scale homogenization problem

We follow the same procedure as for the periodic case. Because the $y$ periodicity is kept in $\mathcal{V}$, the resolution of the homogenized equations is almost the same as in the periodic case. The major difference is that each physical quantity we look for, depends on $\varepsilon_0$ and on $x$.

(i) As for the periodic case, eq. (66) for $i = -1$ and eq. (67) for $i = -1$ give $\sigma^{0,-1,\varepsilon} = 0$ and $u^{0,0,\varepsilon} = [u^{0,0,\varepsilon}](x)$.

(ii) Eq. (66) for $i = -1$ and eq. (67) for $i = 0$ give

$$\nabla_y \cdot \sigma^{0,\varepsilon} = 0,$$

$$\sigma^{0,\varepsilon} = \mu^{0,\varepsilon} \cdot (\nabla_y u^{0,\varepsilon} + \nabla_y u^{0,\varepsilon+1}).$$  \hfill (68)

(69)

Both previous equations lead to this next equality

$$\nabla_y \cdot (\mu^{0,\varepsilon} \cdot \nabla_y u^{0,\varepsilon+1}) = -\nabla_y \cdot (\mu^{0,\varepsilon} \cdot \nabla_y u^{0,\varepsilon}),$$  \hfill (70)

Similarly to the periodic case, we can look for a solution under the form

$$u^{0,1,\varepsilon}(x, y) = [u^{0,1,\varepsilon}](x) + \chi^{0,1,\varepsilon}(x, y) \cdot \nabla_y u^{0,0,\varepsilon}(x).$$  \hfill (71)

As $\sigma^{0,\varepsilon}$ must belong to $\mathcal{V}$, and because $\sigma^{0,\varepsilon}$ is $y$-independent, the first-order corrector $\chi^{0,1,\varepsilon}(x, y)$ (a vector of dimension 2) must also be in $\mathcal{V}$. We will see in Section 3.4, how to proceed to obtain a first-order corrector that has this property. Introducing (71) into (70), we obtain the equations of the cell problems

$$\nabla_y \cdot (\mu^{0,\varepsilon} \cdot (1 + \nabla_y \chi^{0,1,\varepsilon})) = 0.$$  \hfill (72)

To enforce the uniqueness of the solution, we again impose $[\chi^{0,1,\varepsilon}] = 0$.

As in the periodic case, we can rewrite (72) under the equivalent form

$$\nabla_y \cdot H^{0,\varepsilon} = \nabla_y \cdot (\mu^{0,\varepsilon} \cdot G^{0,\varepsilon}) = 0,$$

where $H^{0,\varepsilon}$ is the order 0 homogenized elastic tensor whose components again are (as in the periodic case)

$$\mu^{0,\varepsilon} = [\mu^{0,\varepsilon}] \cdot \nabla \chi^{0,1,\varepsilon}.$$  \hfill (73)

### 3.4 Construction of $\mu^{0,\varepsilon}(x, y)$ and $\sigma^{0,\varepsilon}(x, y)$

Here we are at the heart of the problem: how to conveniently construct $\mu^{0,\varepsilon}(x, y)$ and $\sigma^{0,\varepsilon}(x, y)$, in order for $u^{0,\varepsilon}$, $u^{1,\varepsilon}$ and $\sigma^{0,\varepsilon}$ to be in $\mathcal{V}$? As already said, $\mu^{0,\varepsilon}$ must be built such that the first-order corrector $\chi^{0,1,\varepsilon}$ (eq. 71) and the vector $H^{0,\varepsilon}$ (eq. 77) belong to $\mathcal{V}$. Let
us just recall that \( \chi^{0.1} \) being in \( V \) implies that \( G^{0} \) also is (because it is its gradient). We will show that the reciprocity is verified.

We therefore look for \( \rho^{0}(x, y) \) and \( \mu^{0}(x, y) \) such that

(i) \( \rho^{0} \) and \( H^{0} \) are \( \chi^{0.1} \)-in \( V \);
(ii) \( \rho^{0} \) and \( \mu^{0} \) must be positive definite;
(iii) \( \rho^{0}(x, y) = \rho^{0}(x) \) and \( \mu^{0}(x, x_{0}) = \mu^{0}(x) \).

Let us introduce a starting \( \rho^{0}(x, y) = \rho^{0}(x) \) defined on \( \mathbb{R}^{2} \times y_{s} \) and then periodically extended in \( y \) to \( \mathbb{R}^{2} \). Note that this starting density distribution, is \( x \)-dependent.

The construction of \( \rho^{0}(x, y) \) may then be

\[
\rho^{0}(x, y) = F(\rho^{0}(x, x_{0})) + (\rho^{0} - F(\rho^{0}))(x, y). \tag{83}
\]

Indeed, we have \( \rho^{0}(x, x_{0}) = \rho^{0}(x) \) and \( \rho^{0} \) is in \( T \) by construction. Thanks to (55), \( F(\rho^{0}) = F(\rho^{0}) \). Furthermore, \( F \) being a low-pass filter implies that \( \langle \rho^{0} - F(\rho^{0}) \rangle \) = 0, therefore \( \rho^{0} = F(\rho^{0}) \). Both of these observations lead to \( \rho^{0} = F(\rho^{0}) \), thus \( \rho^{0}(x, y) \) is in \( V \); plus, it is a positive function with a well chosen wavelet \( u_{0} \).

It is important to note that \( \langle \rho^{0} \rangle = F(\rho^{0}) \) is a smooth function of \( x \), as it will be the case for \( \mu^{0} \).

Finding a correct \( \mu^{0} \) is not trivial. We shall follow the same procedure described in Capdeville et al. (2010a) or Capdeville et al. (2010b). We must build \( \mu^{0} \) such that \( G^{0} \) and \( H^{0} \) are in \( V \) and are solutions of

\[
G^{0} = I + V_{x} \chi^{0.1}, \tag{84}
\]

\[
H^{0} = \mu^{0} \cdot G^{0}, \tag{85}
\]

\[
V_{x} \cdot H^{0} = 0, \tag{86}
\]

\[
(G^{0}) = I. \tag{87}
\]

We proceed in the following way

(i) Step 1: a starting \( \mu^{0}(x, y) = \mu^{0}(x) \) is defined on \( \mathbb{R}^{2} \times y_{s} \) and then periodically extended in \( y \) to \( \mathbb{R}^{2} \). Then we solve for the (starting) periodic corrector \( \chi^{0.1}(x, y) \), solution of (86) with periodic boundary conditions on \( y_{s} \).

Then we compute

\[
G^{0}(x, y) = I + V_{x} \chi^{0.1}, \tag{88}
\]

\[
H^{0}(x, y) = \mu^{0} \cdot (x, y) \cdot G^{0}. \tag{89}
\]

At this stage, we can already compute the effective elastic tensor for any \( x \) (see explanation below, after Step 4):

\[
\mu^{0}(x) = \langle F(H^{0}(x, y)) \cdot F(G^{0}(x, y))^{-1} \rangle \tag{90}
\]

(ii) Step 2: now let us compute for any \( y \in y_{x} \).

\[
G^{0}(x, y) = [(G^{0} - F(G^{0}))(x, y)] \cdot F(G^{0}(x, x_{0}))^{-1} + I, \tag{91}
\]

\[
H^{0}(x, y) = [(H^{0} - F(H^{0}))(x, y) + F(H^{0})(x, x_{0})] \cdot F(G^{0})(x, x_{0})^{-1}. \tag{92}
\]

\( G^{0} \) and \( H^{0} \) can be periodically extended from \( y_{x} \) to \( \mathbb{R}^{2} \). We can check that \( H^{0}, G^{0} \in V^{2} \), and that \( G^{0} = I \). Also, according to (76), taking the cell average of (92) indeed leads to the claimed property (90). The last results can be demonstrated the same manner as was done to show that \( \rho^{0} \) indeed belongs to \( V \).

The fact that \( G^{0} = I \) is interesting, because of the definition (84) of \( G^{0} \) and the fact that this quantity belongs to \( V \); it implies that \( V_{x} \chi^{0.1} \) also belongs to \( V \); it has also the following property:

\[
(V_{x} \chi^{0.1}) = 0, \ 	ext{and, because of (61), we finally derive the necessary condition: } \chi^{0.1} \in V. \tag{93}
\]

(iii) Step 3: from (85) we can now build

\[
\mu^{0}(x, y) = \langle H^{0} \cdot (G^{0})^{-1} \rangle (x, y). \tag{94}
\]

(iv) Step 4: once \( \mu^{0}(x, y) \) is known, the whole homogenization procedure can be pursued to find the different components of the first-order corrector vector.

This procedure may seem obscure and here is a tentative of interpretation. The idea behind step one, is to first assume that all scales are fast. Then, the cell problem eq. (72) (which are those of the static equilibrium in mechanics) are solved. They describe the microscopic effect of the imposition of a unit gradient in displacement in the \( y \)-direction: the associated (microscopic) displacement obviously is \( y_{s} = x_{0} \); therefore, \( G^{0} \) only is the gradient of this displacement (as eq. 84 indicates) and \( H^{0} \) the microscopic stress associated to this gradient. The idea behind any elastic homogenization technique, is to find the effective tensor that links an effective stress and an effective deformation: this is exactly what is done in eq. (90) where \( F(H^{0}) \) and \( F(G^{0}) \) play the role of the effective stress and deformation. Eq. (90) allows to directly determine (after filtering) the homogenized elastic tensor, without solving for the `correct’ other quantities, with the starting \( \mu^{0}(x, y) \).

Practically, to ensure the symmetry of this effective elastic tensor, we impose, in this procedure, the equality of \( \mu_{21}^{0} \) and \( \mu_{12}^{0} \), adding then a condition in (90) and solving this latter using a singular value decomposition. Steps 2 and 3 allow to build \( \mu^{0}(x, y) \) by separating the scales of \( H^{0} \) and \( G^{0} \) using the filter \( F \) and making sure that \( G^{0} = I \).

Following all these operations, we have by construction \( \mu^{0}(x, x_{0}) = \mu^{0}(x) \) and \( \mu^{0} \) is positive definite for a well-chosen wavelet \( u_{0} \). We also have checked that \( \chi^{0.1} \) belongs to \( V \) (we have \( G^{0} \in V \) and \( G^{0} = I \) by construction (see Step 2), which, using (61), indeed implies that \( \chi^{0.1} \in \) belongs to \( V \). This corrector is unique for we impose the classical averaging condition \( \chi^{0.1} = 0 \). To determine \( \chi^{0.1} \), we first find \( \mu^{0} \) with (93) and solve (72) once again.

We are not yet able to prove the symmetry of the homogeneous elastic tensor \( \mu^{0} \) in the general case, using the procedure presented in this section. We prove it in the specific case of a non-periodic but layered medium, in Appendix C.

3.5 Homogenization in a non-periodic setting: an example

To validate the previous procedure, we apply it to wave propagation throughout the random model shown in Fig. 5(a). This model is a square of size \( 20 \times 20 \text{ km}^{2} \), surrounded by a 1-km-thick strip of constant physical properties (\( \mu_{11}^{0} = \mu_{22}^{0} = 60 \text{ GPa} \), \( \mu_{33}^{0} = \mu_{33}^{0} = 60 \text{ GPa} \), \( \rho = 2800 \text{ kg m}^{-3} \)), plus an additional PML to avoid reflections at the boundaries in long-time simulations. This square is divided in elements of size \( 100 \times 100 \text{ m}^{2} \). Elastic values in each element are randomly generated (with a uniform distribution) within \( \pm 50 \) per cent of the elastic values of the isotropic surrounding strip.

A reference solution can be computed in this model using SEM. A point source is located at \( S = (x_{S}, x_{S}) = (100 \text{ m}, 10 \text{ km}) \). Its time evolution is described by a Ricker with a central time shift of 0.4 s and a central frequency of 5 Hz (which is associated a corner frequency of about 12.5 Hz). Seismograms are recorded at

© 2010 The Authors, GJI, 182, 1438–1454
Journal compilation © 2010 RAS
As for the homogenization, and practically, we first solve the cell problem (72) on small domains \( y \times x \), with periodic boundary conditions, for a large number of \( x \)-values. We therefore obtain values for first-order correctors and for homogenized coefficients. The resolution of the cell problem is done using a finite element method, based on the same mesh and quadrature as the ones that is used in wave propagation simulations with SEM. In the following example, the polynomial order of the expansion of physical fields, is equal to 4.

In Fig. 5(b) is shown the effective model corresponding to the previous random model Fig. 5(a); it is obtained by applying the homogenization procedure with a low-pass filter whose the cut-off corresponds to a \( \varepsilon_0 \) equal to 0.27. As can be seen, elastic quantities also show rapid spatial variations, but these variations are much smoother than in the original medium. This can be observed more clearly along a section in Fig. 6. It is also possible to determinate the apparent anisotropy, as seen by the wavefield whose the smallest wavelength is much larger than the characteristic size of the (isotropic!) heterogeneities; we measure the apparent anisotropy of the homogenized model as the normalized difference between the homogenized values for \( \mu_{11}^0 \) and \( \mu_{22}^0 : (\mu_{11}^0 - \mu_{22}^0) / \mu_{11}^0 \) (the other terms of the homogenized elastic tensor being very close to be null). As can be seen in Fig. 7, the anisotropy can be quite large, varying between \( -6 \) and \( 6 \) per cent, with (absolute) average values lying between 1 and 2 per cent. In Fig. 8, we compare the particle velocity (recorded at receiver 8, with the source located at S; see Fig. 5a), calculated in the reference medium (grey line with black dots), to the one obtained with the sole order 0 homogenization \( (\mu_{11}^{0,0}) \), for different values of \( \varepsilon_0 \) varying from 2.16 to 0.27 (black line). For large values of \( \varepsilon_0 \), that is, when the homogenized model is too smooth with respect to the minimum wavelength of the wavefield, it appears that the coda of direct waves does not exist or is incorrectly calculated. The more \( \varepsilon_0 \) decreases, the more the coda correctly ‘built’ is: for a \( \varepsilon_0 \) of 0.27, the agreement between the reference and the homogenized solutions, is excellent. Note that the arrival time of the ballistic wave is always correctly predicted, whatever the value of \( \varepsilon_0 \) is; this is not the case when seismograms are recorded in effective media when a simple (and erroneous) filtering is applied to elastic quantities (Fig. 9).

Let us now take a look at the convergence of the homogenized solution \( u^{\varepsilon_0,0} \) towards the reference one \( u^{\text{ref}} \). We introduce the following measure of the error between a given field \( u \) and the reference
Figure 8. Velocity traces computed for the collocated source S, at the receiver 8. The reference solution is in grey with black dots; the order 0 homogenized solution, in black. The $\varepsilon_0$-parameter takes values 2.16 (a), 1.08 (b), 0.54 (c) and 0.27 (d).

Figure 9. Velocity traces recorded at the receiver 8 (Fig. 5a). The reference solution is plotted in grey, and the ‘natural filtered’ solution is in dashed line. Both are clearly out-of-phase.

one, at a given receiver $k$:

$$E_k(\hat{u}) = \frac{\sqrt{\int_0^{20} (\hat{u} - \hat{u}^{\text{ref}})^2(x_k, t)dt}}{\sqrt{\int_0^{20} (\hat{u}^{\text{ref}})^2(x_k, t)dt}},$$

where the maximal bound for the time integral, indicates that seismic recordings of interest last 20 s (after that time the amplitude of the coda wave is completely negligible). Then a combined error $E_{\text{tot}}$ can be defined, taking into account the error at each individual
receiver (Fig. 5a), as

$$E_{\text{un}}(\dot{u}) = \frac{1}{14} \sum_{k=1}^{14} E_k(\dot{u}).$$

(95)

As in the $P$–$SV$ case (Capdeville et al. 2010b), the error $E_{\text{un}}(\dot{u}^{0,0})$ decreases slowly when large $\varepsilon_0$ decreases (Fig. 10). The reason is that the coda only begins to be fully constructed for values of $\varepsilon_0$ around 0.5 (Fig. 8). When $\varepsilon_0 \leq 0.5$, the convergence towards the reference solution is faster than expected: it is in $\varepsilon_0^2$, not in $\varepsilon_0$. For this random example, this observation underlies the fact that high-order terms in the asymptotic expansion (65) are negligible in comparison with the leading term. This is confirmed when calculating the same kind of error, but this time between the partial first-order solution $u^{0,1/2}$ and the reference one (Fig. 10): the effect of the correction term is null for large values of $\varepsilon_0$, and rather weak for small $\varepsilon_0$—even if observable. Note that Capdeville et al. (2010b) pursue the same kind of calculations for smallest values of $\varepsilon_0$, for which the effect of the first-order correction, is more pronounced. A quite good convergence is observed even for high $\varepsilon_0$ values. This is because even if the source is located in a homogeneous zone, the area on which we calculate correctors and homogenized quantities when $\varepsilon_0 = 2.16$, is larger than the homogeneous, surrounding strip of the box model (Fig. 5); thus, the first-order correction for the point source (see Capdeville et al. 2010a) should be taken into account in this case (and the error associated with this large $\varepsilon_0$ should then be lower than the one calculated here).

The effect of the first-order correction can be observed in Fig. 11, where the errors $E(\dot{u}^{0,0})$ and $E(\dot{u}^{0,1/2})$ are reported as a function of $x_1$, along a line joining all receivers plotted on Fig. 5(a). The error decreases when adding the corrective first-order term. We also observe, although it is less marked than in the $P$–$SV$ case, that the error determined for the partial first-order homogenized solution, varies more slowly with $x_1$, than the one for the order 0 solution. This effect can be explained by the fact that the first-order corrector term depends on $\gamma$ (the fast scale)—and therefore ‘corrects’ for errors (between the order 0 homogenized and the reference solution) that vary rapidly in space. This ‘microscopic’ dependence of the first-order correction term can be more precisely observed in Fig. 12, where is plotted this correction ($\dot{u}^{0,1/2} - \dot{u}^{0,0}$) against with the residual $\dot{u}^{0,0} - \dot{u}^{0,0}$ (normalized by the largest value of the velocity in each case), along a line connecting receivers shown in Fig. 5, with $\varepsilon_0 = 0.27$, and at a time $t = 12$ s. The fast oscillations are similar for both curves, and the differences are due to the uncomputed higher-order terms in the asymptotic expansion (65). They are small in amplitude with respect to the one of the reference wavefield (between 1 and 2 per cent of the velocity amplitude), and quite close, which means that the partial first-order homogenized solution we suggested to use in this paper, is an excellent approximation to the wavefield propagating throughout the initial, random model.

4 CONCLUSION

After having recalled classical results of the two-scale homogenization theory in heterogeneous but periodic media, a generalization of this theory has been introduced, as in the companion papers (Capdeville et al. 2010a,b). It is valid for wave propagation in general, non-periodic media—as ones typical of the Earth—when the smallest wavelength of the wavefield, is much larger that the characteristic size of the heterogeneities. A partial first-order homogenized solution can be obtained, adding a first-order correction to the order 0, leading term of the classical (in periodic media) or equivalent (in non-periodic media) asymptotic two-scale expansion used in this theory. It leads to results in good agreement with that calculated in a reference medium (both using the SEM as the wave propagation tool). Although it would be possible to add higher-order terms in the case of periodic media, at the cost of a change in the SEM algorithm, this is not possible in the general case, at less using the theory presented in this paper.

One of the major advantages of a homogenization (or upscaling) procedure, is to obtain effective media (and an associated effective wave equation) smoother than original ones (typically, the Earth’s upper mantle and crust), implying much simpler and sparser meshes.
in numerical simulations and an efficient reduction in calculation time. This smoothing effect in very heterogeneous media, has been shown in both parts of this paper, although we did not insist on benefits relative to the meshing (see Capdeville et al. 2010b, and the application to P–SV propagation in the Marmousi model, which is a good illustration for that issue).

First-order corrections at the source (Capdeville et al. 2010a,b), and boundary conditions’ effects are issues that have not been tackled in this paper; this latter, fundamental when studying surface waves propagation for instance, will be the purpose of a future work, generalizing the results of Capdeville & Marigo (2008).

It is interesting to recall the major result of this paper: as shown by Backus in 1962 for layered media, an isotropic model (at the microscopic scale) is anisotropic when seen by a wavefield whose minimal wavelength is much larger than its characteristic size—and that this macroscopic anisotropy is far from being negligible (around 1 and 2 per cent in average, with peaks at ±6 per cent). This is a serious indication that models of the Earth, as looked for by seismologists when inverting for long-period data, may be considered as anisotropic (then involving, a specific parameterization).

Such a conclusion was shared by Fichtner & Igel (2008), who tried to find an effective, smooth model for the Earth’s crust; their result is even more general because it seems to be valid when the scales of wavelength and of heterogeneities are the same.

Let us finally underline that the homogenization procedure presented in this paper (and specifically, the construction of the fast spatial dependence of the density and elastic tensor, see Section 3.4), should be generalized without any difficulty to three-dimensional models, and that applications to the real Earth are then in sight.
ACKNOWLEDGMENTS

The authors thank Gaetano Festa for kindly providing his 2-D Spectral Element program that was used and modified for this paper. The comments and suggestions of two anonymous reviewers greatly helped to improve the paper. The computations were performed using the Institut de Physique du Globe de Paris (IPGP)’s clusters. This work was funded by the Agence National de la Recherche (ANR) MUSE project under the blanc program.

REFERENCES


APPENDIX A:

Let us recall the classical equations in index notation, and neglecting the source term

\[ \rho^0 \partial_t u_i = \partial_t \sigma_{ij}, \]

\[ \sigma_{ij} = c_{ijkl} \partial_k u_l, \]  \hspace{1cm} (A1)

where the elastic constants satisfy the following symmetries:

\[ c_{ijkl} = c_{ijlk} = c_{jikl}, \]  \hspace{1cm} (A2)

In the antiplane case, the motion is considered as being perpendicular to, let us say, the \((x_1, x_2)\) plane (and parallel to the \(x_3\)-axis), and physical properties only vary in this plane, so the previous equations can be rewritten

\[ \rho^0 \partial_t u_3 = \partial_t \sigma_3, \]

\[ \sigma_3 = c_{333} \partial_t u_3, \]  \hspace{1cm} (A3)

Obviously the constant index can be erased, and we can redefine the elastic tensor, which now depends on four parameters only

\[ \mu_{33} = c_{333}. \]  \hspace{1cm} (A4)

Note that the reduced elastic tensor \(\mu\) is, as \(c\), positive-definite, and that its components are symmetric.

Of course the displacement is a scalar, and the stress, a simple vector. The equations we have to solve are therefore

\[ \rho^0 \partial_t u = \partial_t \sigma, \]

\[ \sigma = \mu \partial_t u. \]  \hspace{1cm} (A5)

APPENDIX B:

We want to show that the equality \(E^{ss} = \langle \rho \chi^1 \rangle\), valid in 1-D (see Capdeville et al. 2010a), can be generalized: \(\mu^{ss} = \langle \rho \chi^2 \rangle\). This can be done in a similar manner, using the property (13) extensively.
Let us start with the differential equation defining $\chi^\varepsilon$:
\[
\partial_i \left( \mu_{ij} \partial_j \chi^\varepsilon \right) = \rho - \langle \rho \rangle .
\]  
(B1)

Multiplying this last equation by $\chi^\varepsilon_i$, taking the cell average of the resulting expression, and using the fact that $\langle \chi^\varepsilon_i \rangle = 0$, we obtain
\[
\langle \chi^\varepsilon_i \partial_j \left( \mu_{ij} \partial_j \chi^\varepsilon \right) \rangle = \langle \rho \chi^\varepsilon_i \rangle .
\]  
(B2)

Using (13), the last equality becomes
\[
- \langle \partial_j \left( \mu_{ij} \partial_j \chi^\varepsilon \right) \rangle = \langle \rho \chi^\varepsilon_i \rangle ,
\]  
(B3)

which can be rewritten
\[
- \langle \partial_j \left( \mu_{ij} \partial_j \chi^\varepsilon \right) \rangle = \langle \partial_j \left( \mu_{ij} \partial_j \chi^\varepsilon \right) \rangle = \langle \partial_j \left( \mu_{ij} \partial_j \chi^\varepsilon \right) \rangle = \langle \rho \chi^\varepsilon_i \rangle .
\]  
(B4)

in which the first term is equal to 0.

Using successively the definition of the first-order corrector (27), and the property (13) the last equality can be transformed as
\[
- \langle \partial_j \left( \mu_{ij} \partial_j \chi^\varepsilon \right) \rangle = \langle \mu_{ij} \partial_j \chi^\varepsilon \rangle = \langle \rho \chi^\varepsilon_i \rangle .
\]  
(B5)

Knowing the definition of $\mu^{\varepsilon}_{ij}$ (eq. 43), the equality $\mu^{\varepsilon}_{ij} = \langle \rho \chi^\varepsilon_i \rangle$ is then proved.

**APPENDIX C:**

We show in this appendix, that in the specific case of a non-periodic, layered medium (that means, one whose properties only change in one specific spatial direction), the effective elastic tensor $\mu^{\varepsilon}_{ij}$ as calculated in (90), is indeed a symmetric tensor.

Let us guess that material properties only evolve in the $x_i(y_1)$-direction, and let us omit the index from now on, as well as the $x$-dependence of physical quantities.

Because all derivatives relative to $y_2$ are null, eq. (72) for the ‘starting’ corrector as defined in the first step of the procedure described in Section 3.4, leads to
\[
\begin{align*}
\partial_y \left( \mu^{\varepsilon}_{11} \chi^\varepsilon \right) & = 0 , \\
\partial_y \left( \mu^{\varepsilon}_{12} + \mu^{\varepsilon}_{21} \partial_y \chi^\varepsilon \right) & = 0 ,
\end{align*}
\]  
(C1)

We can solve both of these equations, for the derivative of the corrector’s components:
\[
\begin{align*}
\partial_y \chi^\varepsilon_{11} & = -1 + \frac{c_1}{\mu^{\varepsilon}_{11}} , \\
\partial_y \chi^\varepsilon_{12} & = - \mu^{\varepsilon}_{21} + \frac{c_1}{\mu^{\varepsilon}_{11}} .
\end{align*}
\]  
(C2)

Both constants $c_1$ and $c_2$ can be calculated by integrating the last equations, over the periodic cell $y_0$ and by using the equalities $\langle \chi^\varepsilon_{11} \rangle = 0$. For instance, $c_1 = 1/2L \int_{-L/2}^{L/2} (1/\mu^{\varepsilon}_{11}) dy$.

Then we can obtain the expressions for $G^{\varepsilon}_{ij}$ and $H^{\varepsilon}_{ij}$, according to (73):
\[
G^{\varepsilon}_{ij} = \begin{pmatrix}
c_1 & c_2 - \mu^{\varepsilon}_{21} \\
0 & 1 \\
\end{pmatrix},
\]  
(C3)

and
\[
H^{\varepsilon}_{ij} = \mu^{\varepsilon}_{ij} - G^{\varepsilon}_{ij} = \begin{pmatrix}
c_1 & c_2 - \mu^{\varepsilon}_{21} \\
\frac{\mu^{\varepsilon}_{21}}{\mu^{\varepsilon}_{11}} & \frac{\mu^{\varepsilon}_{21}}{\mu^{\varepsilon}_{11}} \\
\end{pmatrix}.
\]  
(C4)

The filtered versions of $G^{\varepsilon}_{ij}$ and $H^{\varepsilon}_{ij}$ can now be calculated
\[
F(H^{\varepsilon}_{ij}) = \begin{pmatrix}
c_1 & c_2 - \mu^{\varepsilon}_{21} \\
\frac{\mu^{\varepsilon}_{21}}{\mu^{\varepsilon}_{11}} & \frac{\mu^{\varepsilon}_{21}}{\mu^{\varepsilon}_{11}} \\
\end{pmatrix}.
\]  
(C5)

Inverting the previous matrix is trivial
\[
F(G^{\varepsilon}_{ij})^{-1} = \frac{1}{c_1 F(H^{\varepsilon}_{ij})} \begin{pmatrix}
1 & -c_2 F(H^{\varepsilon}_{ij}) \\
0 & c_1 \\
\end{pmatrix}.
\]  
(C6)

and we are now able to determine the homogenized stiffness tensor, as defined in (90)
\[
\mu^{\varepsilon}_{ij}(x) = F(H^{\varepsilon}_{ij}) \cdot F(G^{\varepsilon}_{ij})^{-1} (x/y_0)
\]  
(C7)

with
\[
\mu^{\varepsilon}_{11} = \frac{1}{\mu^{\varepsilon}_{11}} ,
\]  
(C8)

\[
\mu^{\varepsilon}_{12} = \frac{\mu^{\varepsilon}_{21}}{\mu^{\varepsilon}_{11}},
\]  
(C9)

\[
\mu^{\varepsilon}_{21} = \frac{\mu^{\varepsilon}_{21}}{\mu^{\varepsilon}_{11}},
\]  
(C10)

\[
\mu^{\varepsilon}_{22} = \frac{\mu^{\varepsilon}_{21}}{\mu^{\varepsilon}_{21}}.
\]  
(C11)

One can first notice that, because of the symmetry of $\mu^{\varepsilon}_{ij}$, the homogenized tensor also is symmetric, which is an important, theoretical result: our procedure, in the simplified case of a stratified medium, preserves the symmetry of the elastic tensor.

Secondly, it is interesting to note that each component of the homogenized tensor, is equal to some combinations of filtered quantities only. Furthermore, it appears that these combinations are exactly the same ones, as that obtained analytically for the periodic case (e.g. Cioranescu & Donato 1999). We therefore have generalized this latter, well-known result, to stratified, non-periodic media, the average over the unit cell being replaced by a filtering operation, but on the same quantities.